k-Space Sampling and Fourier Image Reconstruction

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MRI Big Picture

- Pulse sequence
- Spatial information about object is transformed into measured data
- Forward problem

- Computer algorithm
- Measured data is transformed into spatial information about object
- Inverse problem
k-space = spatial Fourier transform

- MRI: spatial encoding using magnetic field gradients

\[ S(t) = \int_{\mathbf{r}} m(\mathbf{r}) e^{j\gamma \mathbf{G}(t) \cdot \mathbf{r}} \, d\mathbf{r} \]

(MR signal after demodulation)

Fourier transform of \( m(\mathbf{r}) \)

Spatial frequency \( k \):

\[ k(t) = \frac{\gamma}{2\pi} \mathbf{G}(t) = \frac{\gamma}{2\pi} \int_{t_0}^{t} \mathbf{G}(\tau) d\tau \]
MR image reconstruction

- Inverse Fourier transform
  - according to the k-space trajectory
k-space trajectories

Homework: Draw the gradient waveforms
Fourier transform

\[ S(k) = \int_{-\infty}^{\infty} s(r) e^{-j2\pi kr} \, dr \]  
(forward)

\[ s(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(k) e^{j2\pi kr} \, dk \]  
(inverse)
Complex exponentials

• Basis functions for the Fourier transform

\[ e^{j2\pi kr} = \cos(2\pi kr) + j\sin(2\pi kr) \]

\[ \text{Im}\left\{ e^{j2\pi kr} \right\} \]

\[ \text{Re}\left\{ e^{j2\pi kr} \right\} \]
Fourier transform properties

- **Linearity:** \[ F\{as_1(r) + bs_2(r)\} = aS_1(k) + bS_2(k) \]

- **Shifting:** \[ F\{s(r - r_0)\} = e^{-j2\pi kr_0} S(k) \]

- **Modulation:** \[ F\{e^{j2\pi k_0 r} s(r)\} = S(k - k_0) \]

- **Conjugate symmetry:** \[ F\{s^*(r)\} = S^*(-k) \]

- **Scaling:** \[ F\{s(ar)\} = \frac{1}{|a|} S\left(\frac{k}{a}\right) \]
Fourier transform properties

• Parseval’s formula:

\[ \int s_1(r)s_2(r) \, dr = \int S_1(k)S_2(k) \, dk \]

\[ s_1 = s_2 = s \quad \Rightarrow \quad \int |s(r)|^2 \, dr = \int |S(k)|^2 \, dk \]

• Convolution & multiplication

\[ F\{s_1(r) * s_2(r)\} = S_1(k)S_2(k) \]

\[ F\{s_1(r)s_2(r)\} = S_1(k) * S_2(k) \]
Fourier transform of basic functions

- **Impulse**: \( \delta(r) \) → \( S(k) \)
- **Boxcar**: \( \sum_{n=-\infty}^{\infty} \delta(r - nT) \) → \( \text{Sinc} \)
- **Sine**: \( \) → \( \frac{1}{T} \)
- **Comb**: \( \sum_{n=-\infty}^{\infty} \delta(r - nT) \) → \( \frac{1}{T} \)
Multidimensional Fourier transform

\[ S(k) = \int_{-\infty}^{\infty} s(r) e^{-j2\pi k \cdot r} \, dr \quad \text{(forward)} \]

\[ s(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(k) e^{j2\pi k \cdot r} \, dk \quad \text{(inverse)} \]

2D \quad r = (x, y) \quad k = (k_x, k_y)

\[ S(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(x, y) e^{-j2\pi (k_x x + k_y y)} \, dx \, dy \]

\[ s(x, y) = \frac{4}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y) e^{j2\pi (k_x x + k_y y)} \, dk_x \, dk_y \]

The multidimensional Fourier transform is separable
2D Fourier transform of basic functions

- **Boxcar**
  - $s(x,y)$
  - $S(k_x,k_y)$
  - $k_0=2$
  - $k_0=4$

- **Sinusoids**
  - $s(x,y)$
  - $S(k_x,k_y)$
  - $k_0=2$
  - $k_0=4$
Analog to digital conversion

- **Nyquist/Shannon theorem**
  - A bandlimited signal with bandwidth B can be reconstructed perfectly from its samples if they are taken at a rate no larger than 1/2B.

\[
\text{Nyquist rate} : \Delta r \leq \frac{1}{2W}
\]

**Homework:** Prove the sampling theorem
Analog to digital conversion

- Nyquist/Shannon theorem

\[ s(n\Delta r) \]

\[ \Delta r \leq \frac{1}{2W} \Rightarrow \text{no aliasing} \]

\[ \Delta r > \frac{1}{2W} \Rightarrow \text{aliasing} \]
Cartesian sampling of k-space

- Equidistant rectilinear sampling
- Bandwidth is defined in the image domain

To avoid aliasing:

$$\Delta k_x \leq \frac{1}{W_x} ; \quad \Delta k_y \leq \frac{1}{W_y}$$
Radial sampling of k-space

To avoid aliasing:

\[
\begin{align*}
\Delta k & \leq \frac{1}{R_r} \\
\Delta \phi & \leq \frac{2\pi}{2(2\pi R_r R_k + 1) + 1}
\end{align*}
\]

\[N_{\text{radial}} = \frac{\pi}{2} N_{\text{Cartesian}}\]
Aliasing examples

\[ \Delta k_y = \frac{2}{W_y} \]

\[ \Delta \phi = 2\Delta \phi_{\text{Nyquist}} \]

\[ \Delta k_y = \frac{4}{W_y} \]

\[ \Delta \phi = 4\Delta \phi_{\text{Nyquist}} \]
Image reconstruction from Fourier samples

\[ \Delta k \leq \frac{1}{W} \Rightarrow \text{no aliasing} \]

• **Perfect reconstruction:**
  – infinite number of k-space samples

• **Practical reconstruction:**
  – finite number of k-space samples
Point spread function (PSF)

- Consequence of finite k-space sampling

\[ \hat{S}(k) = S(k)H(k) \]
\[ s(r) = s(r) * h(r) \]

Point spread function

- Spatial resolution: effective width of the PSF

\[ \Delta r = \frac{1}{h(0)} \int_{-\infty}^{\infty} h(s)ds \]

\[ \Delta r = \text{FWHM}(h(r)) \]
MR image reconstruction

- Cartesian sampling
  - Discrete Fourier transform (DFT)
  - Implementation: fast Fourier transform (FFT)

- Non-Cartesian sampling (radial, spiral, etc)
  - Regridding (general)
  - Backprojection (radial)
Discrete Fourier transform (DFT)

- Discrete signals (sequence of numbers)
- Fast implementation: FFT

\[
S(k) = \sum_{n=0}^{N-1} s(n)e^{-j\frac{2\pi}{N}nk} \quad \text{(forward)}
\]

\[
s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k)e^{j\frac{2\pi}{N}nk} \quad \text{(inverse)}
\]

In Matlab

\[
S=\text{fft}(s)
\]

\[
s=\text{ifft}(S)
\]
Properties of the DFT (FFT)

- DFT is periodic
- Circular convolution: \( (S_1(n) * S_2(n))_N \) \( \iff \) \( S_1(k)S_2(k) \)
- DFT matrix representation: \( S = Fs \)

\[
F = \frac{1}{\sqrt{N}} \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & w & w^2 & \ldots & w^{N-1} \\
1 & w^2 & w^4 & \ldots & w^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{N-1} & w^{2(N-1)} & \ldots & w^{(N-1)(N-1)}
\end{bmatrix}, \text{ where } w = e^{-j\frac{2\pi}{N}}
\]

\( F \) is unitary \( \Rightarrow \quad F^H F = FF^H = I \)
Properties of the DFT (FFT)

• DFT is periodic

• Circular convolution: \( (s_1(n) * s_2(n))_N \iff S_1(k)S_2(k) \)

• DFT matrix representation: \( S = Fs \)
DFT reconstruction of Cartesian k-space data

- \( S(k) \) is known at \( k = n\Delta k \) \( \left( -\frac{N}{2} \leq n \leq \frac{N}{2} \right) \)

\[
\begin{align*}
\text{s} &= \text{fft2}(S) \\
\text{Solution: } r \text{ and } k \text{ from } 0 \text{ to } N-1 \\
\text{s} &= \text{fftshift(fft2(fftshift(S)))}
\end{align*}
\]
DFT reconstruction of Cartesian k-space data

- Spatial resolution

Pixel size

\[ p_x = \frac{W_x}{N_x} \]
\[ p_y = \frac{W_y}{N_y} \]

PSF: sinc function

\[ \Delta x = \frac{1}{h_x(0)} \int_{-\infty}^{\infty} h_x(x)dx = \text{FWHM}(h_x(x)) = \frac{W_x}{N_x} \]

Zero crossing at the center of other pixels!
DFT reconstruction of Cartesian k-space data

- Zero-padding in k-space (Fourier interpolation)
  - Decreases the pixel size but does not increase resolution

\[ p_x = \frac{W_x}{N_{x,\text{padded}}} ; \quad p_y = \frac{W_y}{N_{y,\text{padded}}} \]
DFT reconstruction of Cartesian k-space data

- Gibbs ringing
  - Spurious ringing around sharp edges
  - Caused by k-space truncation
  - Gets stronger for decreasing N

64x64 128x128 256x256 512x512
DFT reconstruction of Cartesian k-space data

• k-space filtering or windowing
  – Reduce Gibbs ringing at the expense of resolution loss

$$S_W(k) = S(k)W(k)$$

• Hamming filter

$$W(k) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N}\right)$$
DFT reconstruction of Cartesian k-space data

- Noise averaging
  - What is the DFT of noise?
  
  Noise:
  \[
  \begin{cases}
  \mu_I = \frac{1}{N} \mu_k = 0 \quad \text{(mean)} \\
  \sigma_I^2 = \frac{1}{N} \sigma_k^2 \quad \text{(variance)}
  \end{cases}
  \]

- Signal to noise ratio (SNR)
  \[
  SNR_I = \frac{S_I}{\sigma_I} = \sqrt{N} SNR_k
  \]

- SNR per pixel
  - Signal per pixel decreases with \(1/N\)
  - Noise std decreases with \(\frac{1}{\sqrt{N}}\)

\[
SNR_{pixel} \propto \frac{1}{\sqrt{N}}
\]